

# On discrete $q$ -ultraspherical polynomials and their duals

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## Abstract

We show that a confluent case of the big  $q$ -Jacobi polynomials  $P_n(x; a, b, c; q) := {}_3\phi_2(q^{-n}, abq^{n+1}, x; aq, cq; q, q)$ , which corresponds to  $a = b = -c$ , leads to a discrete orthogonality relation for imaginary values of the parameter  $a$  (outside of its commonly known domain  $0 < a < q^{-1}$ ). Since  $P_n(x; q^\alpha, q^\alpha, -q^\alpha; q)$  tend to Gegenbauer (or ultraspherical) polynomials in the limit as  $q \rightarrow 1$ , this family represents yet another  $q$ -extension of these classical polynomials, different from the continuous  $q$ -ultraspherical polynomials of Rogers. The dual family with respect to the polynomials  $P_n(x; a, a, -a; q)$  (i.e., the dual discrete  $q$ -ultraspherical polynomials) corresponds to the indeterminate moment problem, that is, these polynomials have infinitely many orthogonality relations. We find orthogonality relations for these polynomials, which have not been considered before. In particular, extremal orthogonality measures for these polynomials are derived.

## 1. Introduction

This paper deals with orthogonality relations for  $q$ -orthogonal polynomials. It is well known that each family of  $q$ -orthogonal polynomials corresponds to determinate or indeterminate moment problem. If a family corresponds to determinate moment problem, then there exists only one positive orthogonality measure  $\mu$  for these polynomials and they constitute a complete orthogonal set in the Hilbert space  $L^2(\mu)$ . If a family corresponds to indeterminate moment problem, then there exists infinitely many orthogonality measures  $\mu$  for these polynomials. Moreover, these measures are divided into two parts: extremal measures and non-extremal measures. If a measure  $\mu$  is extremal, then the corresponding set of polynomials constitute a complete orthogonal set in the Hilbert space  $L^2(\mu)$ . If a measure  $\mu$  is not extremal, then the corresponding family of polynomials is not complete in the Hilbert space  $L^2(\mu)$ .

Importance of orthogonal polynomials and their orthogonality measures stems from the fact that with each family of orthogonal polynomials one can associate a closed symmetric (or self-adjoint) operator  $A$ , representable by a Jacobi matrix. If the corresponding moment problem is indeterminate, then the operator  $A$  is not self-adjoint and has infinitely many self-adjoint extensions. If the operator  $A$  has a physical meaning, then these self-adjoint extensions are especially important. These extensions correspond to extremal orthogonality measures for the same set of polynomials and can be constructed by means of these measures (see, for example, [1], Chapter VII). If the family of polynomials corresponds to determinate moment problem, then the corresponding operator  $A$  is self-adjoint and its spectrum is determined by an orthogonality relation for the polynomials. Moreover, the spectral measure for the operator  $A$  is constructed by means of the orthogonality measure for the corresponding polynomials (see [1], Chapter VII). Thus, orthogonality measures for polynomials are of great interest for applications in operator theory and quantum mechanics.

In the present paper we deal in fact with orthogonality measures for big  $q$ -Jacobi polynomials and their duals. It is well known that the big  $q$ -Jacobi polynomials  $P_n(x; a, b, c; q)$  are orthogonal for values of the parameters in the intervals  $0 < a, b < q^{-1}$ ,  $c < 0$ , and the corresponding moment problem is determinate. We show that these polynomials are also

orthogonal outside of these intervals in a confluent case when  $a = b = -c$ . Since the polynomials  $P_n(x; q^\alpha, q^\alpha, -q^\alpha|q)$  tend to ultraspherical polynomials when  $q \rightarrow 1$ , we call them discrete  $q$ -ultraspherical polynomials (because their orthogonality measure is discrete, contrary to the orthogonality measure for continuous  $q$ -ultraspherical polynomials of Rogers). They correspond to determinate moment problem. We give explicitly an orthogonality relation for  $P_n(x; a, a, -a|q)$  when  $a$  becomes imaginary.

In [2] we introduced a family of  $q$ -orthogonal polynomials  $D_n(\mu(x); a, b, c|q)$ , dual to big  $q$ -Jacobi polynomials (they correspond to indeterminate moment problem), and found an orthogonality relation for them. The corresponding orthogonality measure is not extremal. In the present paper we consider the dual  $q$ -Jacobi polynomials  $D_n(\mu(x); a, b, c|q)$  for  $a = b = -c$ . We find two new orthogonality measures for this case when  $0 < a < q^{-1}$ , which are extremal. Besides, we derive two orthogonality measures for  $D_n(\mu(x); a, a, -a|q)$ , when  $a$  is an imaginary number. These measures are also extremal. We also found infinitely many orthogonality relations for these polynomials, which are difficult to define whether they are extremal or not.

Throughout the sequel we always assume that  $q$  is a fixed positive number such that  $q < 1$ . We use (without additional explanation) notation of the theory of special functions (see, for example, [3] and [4]).

## 2. Big and little $q$ -Jacobi polynomials

The big  $q$ -Jacobi polynomials  $P_n(x; a, b, c; q)$ , introduced by G. E. Andrews and R. Askey [5], are defined by the formula

$$P_n(x; a, b, c; q) := {}_3\phi_2(q^{-n}, abq^{n+1}, x; aq, cq; q, q). \quad (1)$$

The discrete orthogonality relation for these polynomials is

$$\begin{aligned} & r(a, b, c) \sum_{n=0}^{\infty} \frac{(aq, abq/c; q)_n q^n}{(aq/c, q; q)_n} P_m(aq^{n+1}) P_{m'}(aq^{n+1}) \\ & + r(b, a, ab/c) \sum_{n=0}^{\infty} \frac{(bq, cq; q)_n q^n}{(cq/a, q; q)_n} P_m(cq^{n+1}) P_{m'}(cq^{n+1}) \\ & = \frac{(1 - abq)(bq, abq/c, q; q)_m}{(1 - abq^{2m+1})(aq, abq, cq; q)_m} (-ac)^m q^{m(m+3)/2} \delta_{mm'}, \end{aligned} \quad (2)$$

where  $r(a, b, c) := (bq, cq; q)_\infty / (abq^2, c/a; q)_\infty$ . This orthogonality relation holds for  $0 < a, b < q^{-1}$  and  $c < 0$ . For these values of the parameters the big  $q$ -Jacobi polynomials (1) correspond to the determinate moment problem, that is, the orthogonality measure in (2) is unique.

We also need below an explicit form for the little  $q$ -Jacobi polynomials. They are given by the formula

$$p_n(x; a, b|q) := {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qx) \quad (3)$$

(see, for example, (7.3.1) in [3]). The discrete orthogonality relation for the polynomials  $p_m(q^n) \equiv p_m(q^n; a, b|q)$  is of the form

$$\sum_{n=0}^{\infty} \frac{(bq; q)_n (aq)^n}{(q; q)_n} p_m(q^n) p_{m'}(q^n) = \frac{(abq^2; q)_\infty}{(aq; q)_\infty} \frac{(1 - abq)(aq)^m (bq, q; q)_m}{(1 - abq^{2m+1})(abq, aq; q)_m} \delta_{mm'}, \quad (4)$$

where  $0 < a < q^{-1}$  and  $b < q^{-1}$ . For these values of the parameters the little  $q$ -Jacobi polynomials also correspond to the determinate moment problem.

In [2] we discussed two families of orthogonal polynomials, which are dual to big and little  $q$ -Jacobi polynomials, respectively. The dual big  $q$ -Jacobi polynomials  $D_n(\mu(x); a, b, c|q) \equiv D_n(\mu(x; ab); a, b, c|q)$  are defined as

$$D_n(\mu(x; ab); a, b, c|q) := {}_3\phi_2(q^{-x}, abq^{x+1}, q^{-n}; aq, abq/c; q, aq^{n+1}/c), \quad (5)$$

where  $\mu(x; \alpha) := q^{-x} + \alpha q^{x+1}$  represents a  $q$ -quadratic lattice. For  $0 < a, b < q^{-1}$  and  $c < 0$  they satisfy a discrete orthogonality relation with respect to the measure, supported on the points  $\mu(m; ab) = q^{-m} + abq^{m+1}$ ,  $m = 0, 1, 2, \dots$  (see [2], formula (29)). The polynomials  $D_n(\mu(x; ab); a, b, c|q)$  with these values of the parameters correspond to indeterminate moment problem and the orthogonality measure for them in [2] is not extremal.

The dual little  $q$ -Jacobi polynomials  $d_n(\mu(x); a, b|q) \equiv d_n(\mu(x; ab); a, b|q)$  are given by the formula

$$d_n(\mu(x; ab); a, b|q) := {}_3\phi_1(q^{-x}, abq^{x+1}, q^{-n}; bq; q, q^n/a) \quad (6)$$

and they obey the orthogonality relation

$$\sum_{m=0}^{\infty} \frac{(1 - abq^{2m+1})(abq, bq; q)_m}{(1 - abq)(aq, q; q)_m a^{-m} q^{-m^2}} d_n(\mu(m)) d_{n'}(\mu(m)) = \frac{(abq^2; q)_{\infty}}{(aq; q)_{\infty}} \frac{(q; q)_n (aq)^{-n}}{(bq; q)_n} \delta_{nn'} \quad (7)$$

with  $\mu(m) \equiv \mu(m; ab)$ , which is valid for  $0 < a < q^{-1}$  and  $b < q^{-1}$  (see formula (13) in [2]). For these values of the parameters  $a$  and  $b$ , the polynomials (6) correspond to indeterminate moment problem and the orthogonality measure in (7) is extremal.

### 3. Discrete $q$ -ultraspherical polynomials

For the big  $q$ -Jacobi polynomials  $P_n(x; a, b, c; q)$  the following limit relation holds:

$$\lim_{q \uparrow 1} P_n(x; q^{\alpha}, q^{\beta}, -q^{\gamma}; q) = \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)},$$

where  $\gamma$  is real. Therefore,  $\lim_{q \uparrow 1} P_n(x; q^{\alpha}, q^{\alpha}, -q^{\gamma}; q)$  is a multiple of the Gegenbauer (ultraspherical) polynomial  $C_n^{(\alpha-1/2)}(x)$ . For this reason, we introduce the notation

$$C_n^{(a^2)}(x; q) := P_n(x; a, a, -a; q) = {}_3\phi_2(q^{-n}, a^2 q^{n+1}, x; aq, -aq; q, q). \quad (8)$$

It is obvious from (8) that  $C_n^{(a)}(x; q)$  is a rational function in the parameter  $a$ .

From the recurrence relation for the big  $q$ -Jacobi polynomials (1) one readily verifies that the polynomials (8) satisfy the following three-term recurrence relation:

$$x C_n^{(a)}(x; q) = A_n(a) C_{n+1}^{(a)}(x; q) + C_n(a) C_{n-1}^{(a)}(x; q), \quad (9)$$

where  $A_n(a) = (1 - aq^{n+1})/(1 - aq^{2n+1})$ ,  $C_n(a) = 1 - A_n(a)$  and  $C_0^{(a)}(x; q) \equiv 1$ .

An orthogonality relation for  $C_n^{(a)}(x; q)$ , which follows from that for the big  $q$ -Jacobi polynomials and is considered in the next section, holds for positive values of  $a$ . We shall see that the polynomials  $C_n^{(a)}(x; q)$  are orthogonal also for imaginary values of  $a$  and  $x$ . In order to dispense with imaginary numbers in this case, we introduce the following notation:

$$\tilde{C}_n^{(a^2)}(x; q) := (-i)^n C_n^{(-a^2)}(ix; q) = (-i)^n P_n(ix; ia, ia, -ia; q), \quad (10)$$

where  $x$  is real and  $0 < a < \infty$ . The polynomials  $\tilde{C}_n^{(a^2)}(x; q)$  satisfy the recurrence relation

$$x\tilde{C}_n^{(a)}(x; q) = \tilde{A}_n(a) \tilde{C}_{n+1}^{(a)}(x; q) + \tilde{C}_n(a) \tilde{C}_{n-1}^{(a)}(x; q), \quad (11)$$

where  $\tilde{A}_n(a) = A_n(-a) = (1 + aq^{n+1})/(1 + aq^{2n+1})$ ,  $\tilde{C}_n(a) = \tilde{A}_n(a) - 1$ , and  $\tilde{C}_0^{(a)}(x; q) \equiv 1$ . Note that  $\tilde{A}_n(a) \geq 1$  and, therefore, coefficients in the recurrence relation for  $\tilde{C}_n^{(a)}(x; q)$  satisfy the conditions  $\tilde{A}_n(a)\tilde{C}_{n+1}(a) > 0$  of Favard's characterization theorem for  $n = 0, 1, 2, \dots$  (see, for example, [3]). This means that these polynomials are orthogonal with respect to a positive measure with infinitely many points of support. An explicit form of this measure is derived in the next section.

So, we have

$$\tilde{C}_n^{(a)}(x; q) = (-i)^n C_n^{(-a)}(ix; q) = (-i)^n {}_3\phi_2 \left( \begin{matrix} q^{-n}, -aq^{n+1}, ix \\ i\sqrt{aq}, -i\sqrt{aq} \end{matrix} \middle| q, q \right). \quad (12)$$

From the recurrence relation (11) it follows that the polynomials (12) are real for  $x \in \mathbb{R}$  and  $0 < a < \infty$ . From (12) it is also obvious that they are rational functions in the parameter  $a$ .

We show below that the polynomials  $C_n^{(a)}(x; q)$  and  $\tilde{C}_n^{(a)}(x; q)$ , interrelated by (12), are orthogonal with respect to discrete measures. For this reason, they may be regarded as a discrete version of  $q$ -ultraspherical polynomials of Rogers (see, for example, [6]).

**Proposition 1.** *The following expressions for the discrete  $q$ -ultraspherical polynomials (8) hold:*

$$C_{2k}^{(a)}(x; q) = \frac{(q; q^2)_k a^k}{(aq^2; q^2)_k} (-1)^k q^{k(k+1)} p_k(x^2/aq^2; q^{-1}, a|q^2), \quad (13)$$

$$C_{2k+1}^{(a)}(x; q) = \frac{(q^3; q^2)_k a^k}{(aq^2; q^2)_k} (-1)^k q^{k(k+1)} x p_k(x^2/aq^2; q, a|q^2), \quad (14)$$

where  $p_k(y; a, b|q)$  are the little  $q$ -Jacobi polynomials (3).

*Proof.* To start with (13), apply Singh's quadratic transformation for a terminating  ${}_3\phi_2$  series

$${}_3\phi_2 \left( \begin{matrix} a^2, b^2, c \\ abq^{1/2}, -abq^{1/2} \end{matrix} \middle| q, q \right) = {}_3\phi_2 \left( \begin{matrix} a^2, b^2, c^2 \\ a^2b^2q, 0 \end{matrix} \middle| q^2, q^2 \right), \quad (15)$$

which is valid when both sides in (15) terminate (see [3], formula (3.10.13)), to the expression in (8) for the  $q$ -ultraspherical polynomials  $C_{2k}^{(a)}(x; q)$ . This results in the following:

$$C_{2k}^{(a)}(x; q) = {}_3\phi_2 \left( q^{-2k}, aq^{2k+1}, x^2; aq^2, 0; q^2, q^2 \right).$$

Now apply to the above basic hypergeometric series  ${}_3\phi_2$  the transformation formula

$${}_2\phi_1 \left( \begin{matrix} q^{-n}, b \\ c \end{matrix} \middle| q, z \right) = \frac{(c/b; q)_n}{(c; q)_n} {}_3\phi_2 \left( \begin{matrix} q^{-n}, b, bzq^{-n}/c \\ bq^{1-n}/c, 0 \end{matrix} \middle| q, q \right) \quad (16)$$

(see formula (III.7) from Appendix III in [3]) in order to get

$$C_{2k}^{(a)}(x; q) = \frac{(q; q^2)_k a^k}{(aq^2; q^2)_k} (-1)^k q^{k(k+1)} {}_2\phi_1 \left( q^{-2k}, aq^{2k+1}; q; q^2, x^2/a \right).$$

Comparing this formula with the expression for the little  $q$ -Jacobi polynomials (3) one arrives at (13).

One can now prove (14) by induction with the aid of (16) and the recurrence relation (9). Indeed, since  $C_0^{(a)}(x; q) \equiv 1$  and  $A_0 = 1$ , one obtains from (9) that  $C_1^{(a)}(x; q) = x$ . As the next step use the fact that  $C_2^{(a)}(x; q) = {}_3\phi_2(q^{-2}, aq^3, x^2; aq^2, 0; q^2, q^2)$  to evaluate from (9) explicitly that

$$C_3^{(a)}(x; q) = x {}_3\phi_2(q^{-2}, aq^5, x^2; aq^2, 0; q^2, q^2).$$

So, let us suppose that

$$C_{2k-1}^{(a)}(x; q) = x {}_3\phi_2(q^{-2(k-1)}, aq^{2k+1}, x^2; aq^2, 0; q^2, q^2) \quad (17)$$

for  $k = 1, 2, 3, \dots$ , and evaluate a sum  $A_{2k}^{-1} x C_{2k}^{(a)}(x; q) + (1 - A_{2k}^{-1}) C_{2k-1}^{(a)}(x; q)$ . As follows from the recurrence relation (9), this sum should be equal to  $C_{2k+1}^{(a)}(x; q)$ . This is the case because it is equal to

$$\begin{aligned} x \left\{ A_{2k}^{-1} {}_3\phi_2 \left( q^{-2k}, aq^{2k+1}, x^2 \middle| q^2, q^2 \right) + (1 - A_{2k}^{-1}) {}_3\phi_2 \left( q^{-2(k-1)}, aq^{2k+1}, x^2 \middle| q^2, q^2 \right) \right\} \\ = x {}_3\phi_2 \left( q^{-2k}, aq^{2k+3}, x^2 \middle| q^2, q^2 \right), \end{aligned} \quad (18)$$

if one takes into account readily verified identities

$$\begin{aligned} A_{2k}^{-1}(q^{-2k}; q^2)_m + (1 - A_{2k}^{-1})(q^{-2(k-1)}; q^2)_m &= \frac{1 - aq^{2(k+m)+1}}{1 - aq^{2k+1}}(q^{-2k}; q^2)_m, \\ \frac{1 - aq^{2(k+m)+1}}{1 - aq^{2k+1}}(aq^{2k+1}; q^2)_m &= (aq^{2k+3}; q^2)_m, \end{aligned}$$

for  $m = 0, 1, 2, \dots, k$ . The right side of (18) does coincide with  $C_{2k+1}^{(a)}(x; q)$ , defined by the same expression (17) with  $k \rightarrow k+1$ . Thus, it remains only to apply the same transformation formula (16) in order to arrive at (14). Proposition is proved.

*Remark.* Observe that en route to proving formula (14), we established a quadratic transformation

$${}_3\phi_2 \left( q^{-2k-1}, aq^{2k+2}, x \middle| \sqrt{aq}, -\sqrt{aq} \middle| q, q \right) = x {}_3\phi_2 \left( q^{-2k}, aq^{2k+3}, x^2 \middle| q^2, q^2 \right) \quad (19)$$

for the terminating basic hypergeometric polynomials  ${}_3\phi_2$  with  $k = 0, 1, 2, \dots$ . The left side in (19) defines the polynomials  $C_{2k+1}^{(a)}(x; q)$  by (8), whereas the right side follows from the expression (18) for the same polynomials. The formula (19) represents an extension of Singh's quadratic transformation (15) to the case when  $a^2 = q^{-2k-1}$  and, therefore, the left side in (15) terminates, but the right side does not.

It follows from (12)–(14) that

$$\tilde{C}_{2k}^{(a)}(x; q) = \frac{(q; q^2)_k a^k}{(-aq^2; q^2)_k} (-1)^k q^{k(k+1)} p_k(x^2/aq^2; q^{-1}, -a|q^2), \quad (20)$$

$$\tilde{C}_{2k+1}^{(a)}(x; q) = \frac{(q^3; q^2)_k a^k}{(-aq^2; q^2)_k} (-1)^k q^{k(k+1)} x p_k(x^2/aq^2; q, -a|q^2). \quad (21)$$

In particular, it is clear from these formulas that the polynomials  $\tilde{C}_n^{(a)}(x; q)$  are real-valued for  $x \in \mathbb{R}$  and  $a > 0$ .

#### 4. Orthogonality relations for discrete $q$ -ultraspherical polynomials

Since the polynomials  $C_n^{(a)}(x; q)$  are a particular case of the big  $q$ -Jacobi polynomials (as it is obvious from (8)), an orthogonality relation for them follows from (2). Setting  $a = b = -c$ ,  $a > 0$ , into (2) and considering the case when  $m = 2k$  and  $m' = 2k'$ , one verifies that two sums on the left of (2) coincide (since  $ab/c = -a = c$ ) and we obtain the following orthogonality relation for  $C_{2k}^{(a)}(x; q)$ :

$$\sum_{s=0}^{\infty} \frac{(aq^2; q^2)_s q^s}{(q^2; q^2)_s} C_{2k}^{(a)}(\sqrt{a}q^{s+1}; q) C_{2k'}^{(a)}(\sqrt{a}q^{s+1}; q) = \frac{(aq^3; q^2)_{\infty}}{(q; q^2)_{\infty}} \frac{(1-aq)a^{2k}}{1-aq^{4k+1}} \frac{(q; q)_{2k} q^{k(2k+3)}}{(aq; q)_{2k}} \delta_{kk'}, \quad (22)$$

where  $\sqrt{a}$ ,  $a > 0$  denotes a positive value of the root. Thus, *the family of polynomials  $C_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , with  $0 < a < q^{-2}$ , is orthogonal on the set of points  $\sqrt{a}q^{s+1}$ ,  $s = 0, 1, 2, \dots$ .*

As we know, the polynomials  $C_{2k}^{(a)}(x; q)$  are functions in  $x^2$ , that is,  $C_{2k}^{(a)}(\sqrt{a}q^{s+1}; q)$  is in fact a function in  $aq^{2s+2}$ . From (22) it follows that the set of functions  $C_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , constitute a complete basis in the Hilbert space  $\ell^2$  of functions  $f(x^2)$  with the scalar product

$$(f_1, f_2) = \sum_{s=0}^{\infty} \frac{(aq^2; q^2)_s q^s}{(q^2; q^2)_s} f_1(aq^{2s+2}) \overline{f_2(aq^{2s+2})}.$$

This result can be also obtained from the orthogonality relation for the little  $q$ -Jacobi polynomials, if one takes into account formula (13).

Putting  $a = b = -c$ ,  $a > 0$ , into (2) and considering the case when  $m = 2k + 1$  and  $m' = 2k' + 1$ , one verifies that two sums on the left of (2) again coincide and we obtain the following orthogonality relation for  $C_{2k+1}^{(a)}(x; q)$ :

$$\begin{aligned} \sum_{s=0}^{\infty} \frac{(aq^2; q^2)_s q^s}{(q^2; q^2)_s} C_{2k+1}^{(a)}(\sqrt{a}q^{s+1}; q) C_{2k'+1}^{(a)}(\sqrt{a}q^{s+1}; q) \\ = \frac{(aq^3; q^2)_{\infty}}{(q; q^2)_{\infty}} \frac{(1-aq)a^{2k+1}}{(1-aq^{4k+3})} \frac{(q; q)_{2k+1}}{(aq; q)_{2k+1}} q^{(k+2)(2k+1)} \delta_{kk'}. \end{aligned} \quad (23)$$

*The polynomials  $C_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , with  $0 < a < q^{-2}$ , are thus orthogonal on the set of points  $\sqrt{a}q^{s+1}$ ,  $s = 0, 1, 2, \dots$ .*

The polynomials  $x^{-1}C_{2k+1}^{(a)}(x; q)$  are functions in  $x^2$ , that is,  $x^{-1}C_{2k+1}^{(a)}(\sqrt{a}q^{s+1}; q)$  are in fact functions in  $aq^{2s+2}$ . It follows from (23) that the collection of functions  $C_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , constitute a complete basis in the Hilbert space  $\ell^2$  of functions of the form  $F(x) = xf(x^2)$  with the scalar product

$$(F_1, F_2) = \sum_{s=0}^{\infty} \frac{(aq^2; q^2)_s q^s}{(q^2; q^2)_s} F_1(\sqrt{a}q^{s+1}) \overline{F_2(\sqrt{a}q^{s+1})}.$$

Again, this result can be obtained also from the orthogonality relation for the little  $q$ -Jacobi polynomials if one takes into account formula (14).

We have shown that the polynomials  $C_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , as well as the polynomials  $C_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , are orthogonal on the set of points  $\sqrt{a}q^{s+1}$ ,  $s = 0, 1, 2, \dots$ . However, the polynomials  $C_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , are not orthogonal to the polynomials

$C_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , on this set of points. In order to prove that the polynomials  $C_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , are orthogonal to the polynomials  $C_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , one has to take the set of points  $\pm\sqrt{a}q^{s+1}$ ,  $s = 0, 1, 2, \dots$ . Since the polynomials from the first set are even and the polynomials from the second set are odd, for each  $k, k' \in \{0, 1, 2, \dots\}$  the infinite sum

$$I_1 \equiv \sum_{s=0}^{\infty} \frac{(aq^2; q^2)_s q^s}{(q^2; q^2)_s} C_{2k}^{(a)}(\sqrt{a}q^{s+1}; q) C_{2k'+1}^{(a)}(\sqrt{a}q^{s+1}; q)$$

coincides with the following one

$$I_2 \equiv - \sum_{s=0}^{\infty} \frac{(aq^2; q^2)_s q^s}{(q^2; q^2)_s} C_{2k}^{(a)}(-\sqrt{a}q^{s+1}; q) C_{2k'+1}^{(a)}(-\sqrt{a}q^{s+1}; q).$$

Therefore,  $I_1 - I_2 = 0$ . This gives the orthogonality of polynomials from the set  $C_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , with the polynomials from the set  $C_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ . The orthogonality relation for the whole set of polynomials  $C_n^{(a)}(x; q)$ ,  $n = 0, 1, 2, \dots$ , can be written in the form

$$\begin{aligned} & \sum_{s=0}^{\infty} \sum_{\varepsilon=\pm 1} \frac{(aq^2; q^2)_s q^s}{(q^2; q^2)_s} C_n^{(a)}(\varepsilon\sqrt{a}q^{s+1}; q) C_{n'}^{(a)}(\varepsilon\sqrt{a}q^{s+1}; q) \\ &= \frac{(aq^3; q^2)_{\infty}}{(q; q^2)_{\infty}} \frac{(1-aq)a^n}{(1-aq^{2n+1})} \frac{(q; q)_n q^{n(n+3)/2}}{(aq; q)_n} \delta_{nn'}. \end{aligned} \quad (24)$$

We thus conclude that *the polynomials  $C_n^{(a)}(x; q)$ ,  $n = 0, 1, 2, \dots$ , with  $0 < a < q^{-2}$  are orthogonal on the set of points  $\pm\sqrt{a}q^{s+1}$ ,  $s = 0, 1, 2, \dots$ .*

An orthogonality relation for the polynomials  $\tilde{C}_n^{(a)}(x; q)$ ,  $n = 0, 1, 2, \dots$ , is derived by using the relations (20) and (21), as well as the orthogonality relation for the little  $q$ -Jacobi polynomials. Writing down the orthogonality relation (4) for the polynomials  $p_k(x^2/aq^2; q^{-1}, -a|q^2)$  and using the relation (20), one finds an orthogonality relation for the set of polynomials  $\tilde{C}_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ . It has the form

$$\begin{aligned} & \sum_{s=0}^{\infty} \frac{(-aq^2; q^2)_s q^s}{(q^2; q^2)_s} \tilde{C}_{2k}^{(a)}(\sqrt{a}q^{s+1}; q) \tilde{C}_{2k'}^{(a)}(\sqrt{a}q^{s+1}; q) \\ &= \frac{(-aq^3; q^2)_{\infty}}{(q; q^2)_{\infty}} \frac{(1+aq)a^{2k}}{(1+aq^{4k+1})} \frac{(q; q)_{2k}}{(-aq; q)_{2k}} q^{k(2k+3)} \delta_{kk'}. \end{aligned} \quad (25)$$

Consequently, *the family of polynomials  $\tilde{C}_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , is orthogonal on the set of points  $\sqrt{a}q^{s+1}$ ,  $s = 0, 1, 2, \dots$ , for  $a > 0$ .*

As in the case of polynomials  $C_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , the set  $\tilde{C}_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , is complete in the Hilbert space of functions  $f(x^2)$  with the corresponding scalar product.

Similarly, using formula (21) and the orthogonality relation for the little  $q$ -Jacobi polynomials  $p_k(x^2/aq^2; q, -a|q^2)$ , we find an orthogonality relation

$$\begin{aligned} & \sum_{s=0}^{\infty} \frac{(-aq^2; q^2)_s q^s}{(q^2; q^2)_s} \tilde{C}_{2k+1}^{(a)}(\sqrt{a}q^{s+1}; q) \tilde{C}_{2k'+1}^{(a)}(\sqrt{a}q^{s+1}; q) \\ &= \frac{(-aq^3; q^2)_{\infty}}{(q; q^2)_{\infty}} \frac{(1+aq)a^{2k+1}}{(1+aq^{4k+3})} \frac{(q; q)_{2k+1}}{(-aq; q)_{2k+1}} q^{(k+2)(2k+1)} \delta_{kk'} \end{aligned} \quad (26)$$

for the set of polynomials  $\tilde{C}_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ . We see from this relation that for  $a > 0$  the polynomials  $\tilde{C}_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , are orthogonal on the same set of points  $\sqrt{a} q^{s+1}$ ,  $s = 0, 1, 2, \dots$ .

Thus, the polynomials  $\tilde{C}_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , as well as the polynomials  $\tilde{C}_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , are orthogonal on the set of points  $\sqrt{a} q^{s+1}$ ,  $s = 0, 1, 2, \dots$ . However, the polynomials  $\tilde{C}_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , are not orthogonal to the polynomials  $\tilde{C}_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , on this set of points. As in the previous case, in order to prove that the polynomials  $\tilde{C}_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , are orthogonal to the polynomials  $\tilde{C}_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , one has to consider them on the set of points  $\pm \sqrt{a} q^{s+1}$ ,  $s = 0, 1, 2, \dots$ . Since the polynomials from the first set are even and the polynomials from the second set are odd, then the infinite sum

$$I_1 \equiv \sum_{s=0}^{\infty} \frac{(-aq^2; q^2)_s q^s}{(q^2; q^2)_s} \tilde{C}_{2k}^{(a)}(\sqrt{a} q^{s+1}; q) \tilde{C}_{2k+1}^{(a)}(\sqrt{a} q^{s+1}; q)$$

coincides with the sum

$$I_2 \equiv - \sum_{s=0}^{\infty} \frac{(-aq^2; q^2)_s q^s}{(q^2; q^2)_s} \tilde{C}_{2k}^{(a)}(-\sqrt{a} q^{s+1}; q) \tilde{C}_{2k+1}^{(a)}(-\sqrt{a} q^{s+1}; q).$$

Consequently,  $I_1 - I_2 = 0$ . This gives the mutual orthogonality of the polynomials  $\tilde{C}_{2k}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ , to the polynomials  $\tilde{C}_{2k+1}^{(a)}(x; q)$ ,  $k = 0, 1, 2, \dots$ . Thus, the orthogonality relation for the whole set of polynomials  $\tilde{C}_n^{(a)}(x; q)$ ,  $n = 0, 1, 2, \dots$ , can be written in the form

$$\begin{aligned} & \sum_{s=0}^{\infty} \sum_{\varepsilon=\pm 1} \frac{(-aq^2; q^2)_s q^s}{(q^2; q^2)_s} \tilde{C}_n^{(a)}(\varepsilon \sqrt{a} q^{s+1}; q) \tilde{C}_{n'}^{(a)}(\varepsilon \sqrt{a} q^{s+1}; q) \\ &= \frac{(-aq^3; q^2)_{\infty}}{(q; q^2)_{\infty}} \frac{(1+aq) a^n}{(1+aq^{2n+1})} \frac{(q; q)_n}{(-aq; q)_n} q^{n(n+3)/2} \delta_{nn'}. \end{aligned} \quad (27)$$

Note that the family of polynomials  $\tilde{C}_n^{(a)}(x; q)$ ,  $n = 0, 1, 2, \dots$ , corresponds to the determinate moment problem. Thus, the orthogonality measure in (27) is unique. In fact, formula (27) extends the orthogonality relation for the big  $q$ -Jacobi polynomials  $P_n(x; a, a, -a; q)$  to a new domain of values of the parameter  $a$ .

## 5. Dual discrete $q$ -ultraspherical polynomials

The polynomials (5) are dual to the big  $q$ -Jacobi polynomials (1) (see [2]). Let us set  $a = b = -c$  in the polynomials (5), as we made before in the polynomials (1). This gives the polynomials

$$D_n^{(a^2)}(\mu(x; a^2)|q) := D_n(\mu(x; a^2); a, a, -a|q) := {}_3\phi_2 \left( \begin{matrix} q^{-x}, a^2 q^{x+1}, q^{-n} \\ aq, -aq \end{matrix} \middle| q, -q^{n+1} \right), \quad (28)$$

where  $\mu(x; a^2) = q^{-x} + a^2 q^{x+1}$ . We call them dual discrete  $q$ -ultraspherical polynomials. They satisfy the following three-term recurrence relation:

$$\begin{aligned} (q^{-x} + aq^{x+1}) D_n^{(a)}(\mu(x; a)|q) &= -q^{-2n-1} (1 - aq^{2n+2}) D_{n+1}^{(a)}(\mu(x; a)|q) \\ &+ q^{-2n-1} (1 + q) D_n^{(a)}(\mu(x; a)|q) - q^{-2n} (1 - q^{2n}) D_{n-1}^{(a)}(\mu(x; a)|q). \end{aligned}$$



For the polynomials  $D_n^{(a^2)}(\mu(x; a^2)|q)$  with imaginary  $a$  we introduce the notation

$$\tilde{D}_n^{(a^2)}(\mu(x; -a^2)|q) := D_n(\mu(x; -a^2); ia, ia, -ia|q) := {}_3\phi_2 \left( \begin{matrix} q^{-x}, -a^2 q^{x+1}, q^{-n} \\ iaq, -iaq \end{matrix} \middle| q, -q^{n+1} \right). \quad (29)$$

The polynomials  $\tilde{D}_n^{(a)}(\mu(x; -a^2)|q)$  satisfy the recurrence relation

$$\begin{aligned} (q^{-x} - aq^{x+1})\tilde{D}_n^{(a)}(\mu(x; -a)|q) &= -q^{-2n-1}(1 + aq^{2n+2})\tilde{D}_{n+1}^{(a)}(\mu(x; -a)|q) \\ &+ q^{-2n-1}(1 + q)\tilde{D}_n^{(a)}(\mu(x; -a)|q) - q^{-2n}(1 - q^{2n})\tilde{D}_{n-1}^{(a)}(\mu(x; -a)|q). \end{aligned}$$

It is obvious from this relation that the polynomials  $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$  are real for  $x \in \mathbb{R}$  and  $a > 0$ . For  $a > 0$  these polynomials satisfy the conditions of Favard's theorem and, therefore, are orthogonal.

**Proposition 2.** *The following expressions for the dual discrete  $q$ -ultraspherical polynomials (28) hold:*

$$D_n^{(a)}(\mu(2k; a)|q) = d_n(\mu(k; q^{-1}a); q^{-1}, a|q^2) = {}_3\phi_1 \left( \begin{matrix} q^{-2k}, aq^{2k+1}, q^{-2n} \\ aq^2 \end{matrix} \middle| q^2, q^{2n+1} \right), \quad (30)$$

$$D_n^{(a)}(\mu(2k+1; a)|q) = q^n d_n(\mu(k; qa); q, a|q^2) = q^n {}_3\phi_1 \left( \begin{matrix} q^{-2k}, aq^{2k+3}, q^{-2n} \\ aq^2 \end{matrix} \middle| q^2, q^{2n-1} \right), \quad (31)$$

where  $d_n(\mu(x; bc); b, c|q)$  are the dual little  $q$ -Jacobi polynomials (6).

*Proof.* Applying to the right side of (28) the formula (III.13) from Appendix III in [3] and then Singh's quadratic relation (15) for terminating  ${}_3\phi_2$  series, after some transformations one obtains

$$D_n^{(a^2)}(\mu(2k; a^2)|q) = a^{-2k} q^{-k(2k+1)} {}_3\phi_2 \left( \begin{matrix} q^{-2k}, a^2 q^{2k+1}, a^2 q^{2n+2} \\ a^2 q^2, 0 \end{matrix} \middle| q^2, q^2 \right).$$

Now apply the relation (0.6.26) from [7] in order to get

$$D_n^{(a^2)}(\mu(2k; a^2)|q) = \frac{(q^{-2k+1}; q^2)_k}{(a^2 q^2; q^2)_k} {}_2\phi_1 \left( \begin{matrix} q^{-2k}, a^2 q^{2k+1} \\ q \end{matrix} \middle| q^2, q^{2n+2} \right).$$

Using the formula (III.8) from [3], one arrives at the expression for  $D_n^{(a^2)}(\mu(2k; a^2)|q)$  in terms of the basic hypergeometric function from (30), coinciding with  $d_n(\mu(k; q^{-1}a^2); q^{-1}, a^2|q^2)$ .

The formula (31) is proved in the same way by using the relation (19). Proposition is proved.

For the polynomials  $\tilde{D}_n^{(a)}(\mu(m; -a)|q)$  we have the expressions

$$\tilde{D}_n^{(a)}(\mu(2k; -a)|q) = d_n(\mu(k; -q^{-1}a); q^{-1}, -a|q^2) = {}_3\phi_1 \left( \begin{matrix} q^{-2k}, -aq^{2k+1}, q^{-2n} \\ -aq^2 \end{matrix} \middle| q^2, q^{2n+1} \right), \quad (32)$$

$$\tilde{D}_n^{(a)}(\mu(2k+1; -a)|q) = q^n d_n(\mu(k; -qa); q, -a|q^2) = q^n {}_3\phi_1 \left( \begin{matrix} q^{-2k}, -aq^{2k+3}, q^{-2n} \\ -aq^2 \end{matrix} \middle| q^2, q^{2n-1} \right). \quad (33)$$

It is plain from the explicit formulas that the polynomials  $D_n^{(a)}(\mu(x; a)|q)$  and  $\tilde{D}_n^{(a)}(\mu(x; a)|q)$  are rational functions of  $a$ .

## 6. Orthogonality relations for dual discrete $q$ -ultraspherical polynomials

An example of the orthogonality relation for  $D_n^{(a^2)}(\mu(x; a^2)|q) \equiv D_n(\mu(x; a^2); a, a, -a|q)$ ,  $0 < a < q^{-1}$ , has been discussed in [2]. However, these polynomials correspond to indeterminate moment problem and, therefore, this orthogonality relation is not unique. Let us find another orthogonality relations. In order to derive them we take into account the relations (30) and (31), and the orthogonality relation (7) for the dual little  $q$ -Jacobi polynomials. By means of formula (30), we arrive at the following orthogonality relation for  $0 < a < q^{-2}$ :

$$\sum_{k=0}^{\infty} \frac{(1 - aq^{4k+1})(aq; q)_{2k}}{(1 - aq)(q; q)_{2k}} q^{k(2k-1)} D_n^{(a)}(\mu(2k)|q) D_{n'}^{(a)}(\mu(2k)|q) = \frac{(aq^3; q^2)_{\infty}}{(q; q^2)_{\infty}} \frac{(q^2; q^2)_n q^{-n}}{(aq^2; q^2)_n} \delta_{nn'},$$

where  $\mu(2k) \equiv \mu(2k; a)$ . The relation (31) leads to the orthogonality relation, which can be written in the form

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(1 - aq^{4k+3})(aq; q)_{2k+1}}{(1 - aq)(q; q)_{2k+1}} q^{k(2k+1)} D_n^{(a)}(\mu(2k+1)|q) D_{n'}^{(a)}(\mu(2k+1)|q) \\ = \frac{(aq^3; q^2)_{\infty}}{(q; q^2)_{\infty}} \frac{(q^2; q^2)_n q^{-n}}{(aq^2; q^2)_n} \delta_{nn'}, \end{aligned}$$

where  $\mu(2k+1) \equiv \mu(2k+1; a)$  and  $0 < a < q^{-2}$ .

Thus, we have obtained two orthogonality relations for the polynomials  $D_n^{(a)}(\mu(x; a)|q)$ ,  $0 < a < q^{-2}$ , one on the lattice  $\mu(2k; a) \equiv q^{-2k} + aq^{2k+1}$ ,  $k = 0, 1, 2, \dots$ , and another on the lattice  $\mu(2k+1; a) \equiv q^{-2k-1} + aq^{2k+3}$ ,  $k = 0, 1, 2, \dots$ . The corresponding orthogonality measures are extremal since they are extremal for the dual little  $q$ -Jacobi polynomials from formulas (30) and (31) (see [2]).

The polynomials  $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$  also correspond to indeterminate moment problem and, therefore, they have infinitely many positive orthogonality measures. Some of their orthogonality relations can be derived in the same manner as for the polynomials  $D_n^{(a)}(\mu(x)|q)$  by using the connection (32) and (33) of these polynomials with the dual little  $q$ -Jacobi polynomials (6). The relation (32) leads to the orthogonality relation

$$\sum_{k=0}^{\infty} \frac{(1 + aq^{4k+1})(-aq; q)_{2k}}{(1 + aq)(q; q)_{2k}} q^{k(2k-1)} \tilde{D}_n^{(a)}(\mu(2k)|q) \tilde{D}_{n'}^{(a)}(\mu(2k)|q) = \frac{(-aq^3; q^2)_{\infty}}{(q; q^2)_{\infty}} \frac{(q^2; q^2)_n q^{-n}}{(-aq^2; q^2)_n} \delta_{nn'},$$

where  $\mu(2k) \equiv \mu(2k; -a)$ , and the relation (33) gives rise to the orthogonality relation, which can be written in the form

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(1 + aq^{4k+3})(-aq; q)_{2k+1}}{(1 + aq)(q; q)_{2k+1}} q^{k(2k+1)} \tilde{D}_n^{(a)}(\mu(2k+1)|q) \tilde{D}_{n'}^{(a)}(\mu(2k+1)|q) \\ = \frac{(-aq^3; q^2)_{\infty}}{(q; q^2)_{\infty}} \frac{(q^2; q^2)_n q^{-n}}{(-aq^2; q^2)_n} \delta_{nn'}, \end{aligned}$$

where  $\mu(2k+1) \equiv \mu(2k+1; -a)$ . In both cases,  $a$  is any positive number.

Thus, in the case of the polynomials  $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$  we also have two orthogonality relations. *The corresponding orthogonality measures are extremal* since they are extremal for the dual little  $q$ -Jacobi polynomials from formulas (32) and (33).

Note that the extremal measures for the polynomials  $D_n^{(a)}(\mu(x)|q)$  and  $\tilde{D}_n^{(a)}(\mu(x)|q)$ , discussed in this section, can be used for constructing self-adjoint extensions of the closed symmetric operators, connected with the three-term recurrence relations for these polynomials and representable in an appropriate basis by a Jacobi matrix (details of such construction are given in [1], Chapter VII). These operators are representation operators for discrete series representations of the quantum algebra  $U_q(\mathfrak{su}_{1,1})$ . Moreover, the parameter  $a$  for these polynomials is connected with the number  $l$ , which characterizes the corresponding representation  $T_l$  of the discrete series.

## 7. Other orthogonality relations

The polynomials  $D_n^{(a)}(\mu(x; a)|q)$  and the polynomials  $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$  correspond to indeterminate moment problems. For this reason, there exist infinitely many orthogonality relations for them. Let us derive some of these relations for the polynomials  $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$ , by using the orthogonality relations for the polynomials (5.18) in [8], which are (up to a factor) of the form

$$u_n(\sinh \xi; t_1, t_2|q) = {}_3\phi_1 \left( \begin{matrix} q e^\xi/t_1, -q e^{-\xi}/t_1, q^{-n} \\ -q^2/t_1 t_2 \end{matrix} \middle| q, q^n t_1/t_2 \right) \quad (34)$$

and a one-parameter family of orthogonality relations for them, characterized by a number  $d$ ,  $q \leq d < 1$ , are given by the formula

$$\sum_{n=-\infty}^{\infty} \frac{(-t_1 q^{-n}/d, t_1 q^n d, -t_2 q^{-n}/d, t_2 q^n d; q)_{\infty}}{(-t_1 t_2/q; q)_{\infty}} \frac{d^{4n} q^{n(2n-1)} (1 + d^2 q^{2n})}{(-d^2; q)_{\infty} (-q/d^2; q)_{\infty} (q; q)_{\infty}} \\ \times u_r((d^{-1} q^{-n} - d q^n)/2; t_1, t_2|q) u_s((d^{-1} q^{-n} - d q^n)/2; t_1, t_2|q) = \frac{(q; q)_r (t_1/t_2)^r}{(-q^2/t_1 t_2; q)_r q^r} \delta_{rs}. \quad (35)$$

The orthogonality measure here is positive for  $t_1, t_2 \in \mathbb{R}$  and  $t_1 t_2 > 0$ . It is not known whether these measures are extremal or not.

In order to use the orthogonality relation (35) for the polynomials  $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$ , let us consider the transformation formula

$${}_3\phi_2 \left( \begin{matrix} q^{-2k}, -a^2 q^{2k+1}, q^{-n} \\ iaq, -iaq \end{matrix} \middle| q, -q^{n+1} \right) = {}_3\phi_1 \left( \begin{matrix} q^{-2k}, -a^2 q^{2k+1}, q^{-2n} \\ -a^2 q^2 \end{matrix} \middle| q^2, q^{2n+1} \right), \quad (36)$$

which is true for any nonnegative integer values of  $k$ . This formula is obtained by equating two expressions (29) and (32) for the dual discrete  $q$ -ultraspherical polynomials  $\tilde{D}_n^{(a)}(\mu(2k; -a)|q)$ . Observe that (36) is still valid if one replaces the numerator parameters  $q^{-2k}$  and  $-a^2 q^{2k+1}$  in both sides of it by  $c^{-1} q^{-2k}$  and  $-ca^2 q^{2k+1}$ ,  $c \in \mathbb{C}$ , respectively. Indeed, the left side of (36) represents a finite sum:

$${}_3\phi_2(q^{-n}, \alpha, \beta; \gamma, \delta; q, z) := \sum_{m=0}^n \frac{(q^{-n}, \alpha, \beta; q)_m}{(\gamma, \delta, q; q)_m} z^m. \quad (37)$$

In the case in question  $\alpha = q^{-2k}$  and  $\beta = -a^2 q^{2k+1}$ , so the  $q$ -shifted factorial  $(\alpha, \beta; q)_m$  in (37) is equal to

$$(q^{-2k}, -a^2 q^{2k+1}; q)_m = \prod_{j=0}^{m-1} [1 - a^2 q^{2j+1} - q^j (q^{-2k} - a^2 q^{2k+1})] = \prod_{j=0}^{m-1} [1 - a^2 q^{2j+1} - q^j \mu(2k; -a^2)], \quad (38)$$

where, as before,  $\mu(2k; -a^2) = q^{-2k} - a^2 q^{2k+1}$ . The left side in (36) thus represents a polynomial  $p_n(x)$  in the  $\mu(2k; -a^2)$  of degree  $n$ . In a similar manner, one easily verifies that the right side of (36) also represents a polynomial  $p'_n(x)$  of degree  $n$  in the same variable  $\mu(2k; -a^2)$ . In other words, the transformation formula (36) states that the polynomials  $p_n(x)$  and  $p'_n(x)$  are equal to each other on the infinite set of distinct points  $x_k = \mu(2k; -a^2)$ ,  $k \geq n$ . Hence, they are identical.

But the point is that

$$\begin{aligned} (c^{-1}q^{-2k}, -ca^2q^{2k+1}; q)_m &= \prod_{j=0}^{m-1} [1 - a^2q^{2j+1} - q^j(c^{-1}q^{-2k} - ca^2q^{2k+1})] \\ &= \prod_{j=0}^{m-1} [1 - a^2q^{2j+1} - q^j\mu_c(2k; -a^2)], \end{aligned}$$

where  $\mu_c(2k; -a^2) = c^{-1}q^{-2k} - ca^2q^{2k+1}$ . So, the replacements  $q^{-2k} \rightarrow c^{-1}q^{-2k}$  and  $a^2q^{2k+1} \rightarrow ca^2q^{2k+1}$  change only the variable,  $\mu(2k; -a^2) \rightarrow \mu_c(2k; -a^2)$ , whereas all other factors in both sides of (36) are unaltered. Thus, our statement becomes evident.

We are now in a position to establish other orthogonality relations for the polynomials  $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$ , different from those, obtained in section 6. To achieve this, we use the fact that the polynomials  $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$  at the points  $x_k^{(d)} := 2k + \ln(\sqrt{aq}/d)/\ln q$  are equal to

$$\tilde{D}_n^{(a)}(\mu(x_k^{(d)}; -a)|q) = {}_3\phi_2 \left( \begin{matrix} q^{-2k}d^{-1}\sqrt{aq}, -q^{2k}d\sqrt{aq}, q^{-n} \\ i\sqrt{a}q, -i\sqrt{a}q \end{matrix} \middle| q, -q^{n+1} \right), \quad (39)$$

where  $\mu(x_k^{(d)}; -a) = \sqrt{aq}(d^{-1}q^{-2k} - dq^{2k})$ . From (34) and (36) it then follows that

$$\tilde{D}_n^{(a)}(\mu(x_k^{(d)}; -a)|q) = u_n \left( (d^{-1}q^{-2k} - dq^{2k})/2; \sqrt{q^3/a}, \sqrt{q/a} | q^2 \right).$$

Hence, from the orthogonality relations (35) one obtains infinite number of orthogonality relations for the polynomials  $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$ , which are parametrized by the same  $d$  as in (35). They are of the form

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-t_1q^{-2n}/d, t_1q^{2n}d, -t_2q^{-2n}/d, t_2q^{2n}d; q^2)_{\infty}}{(-t_1t_2/q^2; q^2)_{\infty}} \frac{d^{4n}q^{2n(2n-1)}(1 + d^2q^{4n})}{(-d^2; q^2)_{\infty}(-q^2/d^2; q^2)_{\infty}(q^2; q^2)_{\infty}} \\ \times \tilde{D}_r^{(a)}(\mu(x_n^{(d)}; -a)|q) \tilde{D}_s^{(a)}(\mu(x_n^{(d)}; -a)|q) = \frac{(q^2; q^2)_r}{(-q^2a; q^2)_r^2} \delta_{rs}, \end{aligned} \quad (40)$$

where  $t_1 = \sqrt{q^3/a}$  and  $t_2 = \sqrt{q/a}$ .

It is important to know whether an orthogonality measure for a family of polynomials is extremal or not. The extremality of the measures in (40) for the polynomials  $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$  depends on the extremality of the orthogonality measures in (35) for the polynomials (34). If any of the measures in (35) is extremal, then the corresponding measure in (40) is also extremal.

## 8. Concluding remarks

The Askey scheme [7] conveniently embraces, up to  ${}_4\phi_3$ -level, all known families of orthogonal basic hypergeometric polynomials: from continuous  $q$ -Hermite, Stieltjes–Wigert,

and discrete  $q$ -Hermite polynomials on the ground level of this scheme (for these families do not contain any parameter other than  $q$ ) up to the four-parameter Askey–Wilson and  $q$ -Racah polynomials on the highest, fourth level. The members of this hierarchy are known to possess the simple property: zero values and limit cases of the parameters for any  $q$ -family lead to other sets in the same hierarchy and does not yield anything novel. The situation seems to be different for confluent cases of parameters. So in the present paper we have studied a confluent case for the big  $q$ -Jacobi polynomials  $P_n(x; a, b, c; q)$  when  $a = b = -c$ . It turns out that the emerging one-parameter family of  $q$ -polynomials represents a  $q$ -extension of Gegenbauer (or ultraspherical) polynomials with a discrete orthogonality relation with respect to the measure, supported on the infinite set of points  $\pm a^{1/2} q^{k+1}$ ,  $k = 0, 1, 2, \dots$ . This family is different from all known one-parameter sets, which occupy the first level in the Askey  $q$ -scheme.

The discrete  $q$ -ultraspherical polynomials and their duals are evidently interesting by themselves. Besides, this particular example, considered by us, demonstrates that other confluent cases from the Askey  $q$ -scheme certainly deserve our close attention.

Another instance of similar interest is provided by the  $q$ -Meixner–Pollaczek polynomials

$$P_n(x; a|q) := e^{-in\phi} \frac{(a^2; q)_n}{a^n (q; q)_n} {}_3\phi_2 \left( \begin{matrix} q^{-n}, ae^{i(\theta+2\phi)}, ae^{-i\theta} \\ a^2, 0 \end{matrix} \middle| q, q \right), \quad x = \cos(\theta + \phi).$$

They are orthogonal on the interval  $-\pi \leq \theta \leq \pi$  for  $0 < a < 1$  (see, for example, section 3.9 in [7]). But let us replace  $a$  by  $ia$  and assume that  $\phi = -\pi/2$ . One obtains then real polynomials

$$\tilde{P}_n(\sin \theta; a|q) := \frac{(-a^2; q)_n}{a^n (q; q)_n} {}_3\phi_2 \left( \begin{matrix} q^{-n}, iae^{-i\theta}, -iae^{i\theta} \\ -a^2, 0 \end{matrix} \middle| q, q \right), \quad (42)$$

which satisfy the three-term recurrence relation

$$2x\tilde{P}_n(x; a|q) = (1 - q^{n+1})\tilde{P}_{n+1}(x; a|q) + (1 + a^2q^{n-1})\tilde{P}_{n-1}(x; a|q).$$

It is now obvious that the polynomials (42) satisfy the conditions  $A_n C_{n+1} > 0$ ,  $n = 0, 1, 2, \dots$ , of Favard's theorem for arbitrary real  $a$  and, therefore, they are orthogonal. An orthogonality relation has the form

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{P}_m(\sin \theta; a|q) \tilde{P}_n(\sin \theta; a|q) w(\theta) d\theta = (q; q)_n^{-1} (q, -a^2 q^n; q)_{\infty}^{-1} \delta_{mn},$$

where

$$w(\theta) = |(-e^{2i\theta}; q)_{\infty} / (-a^2 e^{2i\theta}; q^2)_{\infty}|^2.$$

A more detailed discussion of this orthogonality property will be given elsewhere.

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